Automorphism Groups of the Pancake Graphs *

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Abstract

It is well-known that the pancake graphs are widely used as models for interconnection networks [1]. In this paper, some properties of the pancake graphs are investigated. We first prove that the pancake graph, denoted by P_n ($n \geq 4$), is super-connected and hyper-connected. Further, we study the symmetry of P_n and completely determine its full automorphism group, which shows that P_n ($n \geq 5$) is a graphical regular representation of S_n.

Key words: Interconnection networks; pancake graph; super-connected; hyper-connected; efficient dominating sets; automorphism group.

AMS Classifications: 05C25, 05C69

1 Introduction

For a simple graph Γ, we denote its vertex set, edge set and full automorphism group respectively by $V(\Gamma), E(\Gamma)$ and $\mathrm{Aut}(\Gamma)$. Γ is said to be *vertex-transitive* or *edge-transitive* if $\mathrm{Aut}(\Gamma)$ acts transitively on $V(\Gamma)$ or $E(\Gamma)$, respectively. Let G be a finite group and S a subset of G not

*This work is supported by National Natural Science Foundation of China (No:10971137), the National Basic Research Program (973) of China (No.2006CB805900), and a grant of Science and Technology Commission of Shanghai Municipality (STCSM, No: 09XD1402500) .

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containing the identity element 1 with $S = S^{-1}$. The *Cayley graph* $\Gamma := \mathrm{Cay}(G, S)$ on G with respect to S is defined by

$$V(\Gamma)=G,\ E(\Gamma)=\{(g, gs) : g\in G,\ s\in S\}.$$

Clearly, Γ is a $|S|$-regular and vertex-transitive graph, since $\mathrm{Aut}(\Gamma)$ contains the left regular representation $L(G)$ of G. Moreover, Γ is connected if and only if G is generated by S.

A *permutation* σ on the set $X = \{1, 2, \cdots, n\}$ is a bijective mapping from X to X. As usual, we denote by S_n the group of all permutations on X, which is called the *symmetric group*. The *pancake graph* P_n, also called the *prefix-reversal graph* is the Cayley graph $\mathrm{Cay}(S_n, PR_n)$, where $PR_n = \{r_{1j} : 2 \le j \le n\}$ and $r_{1j} = \begin{pmatrix} 1 & 2 & \cdots & j & j+1 & \cdots & n \\ j & j-1 & \cdots & 1 & j+1 & \cdots & n \end{pmatrix}$.

The pancake graph is well-known because of the famous unsolved combinatorial problem about computing its diameter, which has been introduced by [6], and has been studied in several papers[8, 11, 12]. The pancake graph was often used as a model for interconnection networks of parallel computers [1] due to its attractive properties regarding degree, diameter, symmetry, embeddings and self similarity. The pancake graph P_n corresponds to the n-dimensional pancake network in computer science such that this network has processors labeled by permutations on X and two processors are connected when the label of one is obtained from the other by some $r_{1j}, 2 \le j \le n$. The diameter of this network corresponds to the worst communication delay for transmitting information in a system. Morover, many researchers (see [13, 16, 21]) have investigated some other properties of P_n, such as the hamilton-connectedness, cycle-embedding problem, super-connectivity.

A graph X is said to be *super-connected* [4] if each minimum vertex cut is the neighbor set of a single vertex in X. A graph X is said to be *hyper-connected* [10] if for every minimum vertex cut D of X, $X - D$ has exactly two components, one of which is an isolated vertex. In [15], Li investigated the super-connectedness and hyper-connectedness of the reversal Cayley graph and pointed out that it is unknown for the pancake graph. Here we solve this problem and prove that the pancake graph P_n ($n \ge 4$) is super-connected and hyper-connected.

An independent set D of vertices in a graph is called an *efficient dominating set* [2, 3] if each vertex not in D is adjacent to exactly one vertex in D. In [5], Dejter investigated the efficient dominating sets of Cayley graphs on the symmetric groups, which implied that there exists the efficient dominating sets in the pancake graph. In addtion, the efficient dominating sets are used in optimal broadcasting algorithms for multiple messages on the pancake graphs [20]. Motivated by these results, we completely characterize all the efficient dominating sets in P_n ($n \ge 3$).

A graph $\Gamma = (V, E)$ is a *graphical regular representation (GRR)*[19] of the finite group G if $\mathrm{Aut}(\Gamma) = G$ and $\mathrm{Aut}(\Gamma)$ acts regularly on V. It is well-known that for the interconnection networks modeled by Cayley graphs, the symmetry is one of the problems focused by many researchers. In [14], Lakshmivarahan investigated the symmetry of the pancake graph and showed that P_n is not edge-transitive and hence not distance-transitive. In this paper, we further study the symmetry of P_n and completely determine the automorphism group of P_n, which shows that P_n ($n \geq 5$) is a graphical regular representation of S_n and hence not edge-transitive and distance-transitive.

The rest part of this paper is organized as follows. In Section 2, we first prove that the pancake graph P_n ($n \geq 4$) is super-connected and hyper-connected, then we show that there are exactly n efficient dominating sets $B^{(i)}$ ($i = 1, 2, \cdots, n$) in P_n ($n \geq 3$), where $B^{(i)} = \{\pi \in S_n : \pi(1) = i\}$. In section 3, we prove that the full automorphism groups of P_n ($n \geq 5$) is the left regular representation of S_n, i.e. $\mathrm{Aut}(P_n) = L(S_n)$.

2 Some properties of P_n

In table 1 of [15], it has been pointed out that the super-connectedness and hyper-connectedness of P_n are unknown. In this section, we first prove that the pancake graph P_n ($n \geq 4$) is super-connected and hyper-connected. Following [15], we introduce some notations and terminologies. Let X be a graph and F a subset of $V(X)$. Set $N(F) = \{x \in V(X) \setminus F : \exists y \in F, s.t. xy \in E(X)\}$, $C(F) = F \cup N(F)$, $R(F) = V(X) \setminus C(F)$. A subset $F \subseteq V(X)$ is a *fragment* if $|N(F)| = \kappa(X)$ and $R(F) \neq \emptyset$, where $\kappa(X)$ is the vertex-connectivity of X. A fragment F with $2 \leq |F| \leq |V(X)| - \kappa(X) - 2$ is called a *strict fragment*. A strict fragment with minimum cardinality is called a *superatom*.

The following result is due to Mader [17]:

Lemma 2.1 [17] *If X is a connected undirected graph which is a vertex-transitive and K_4-free, then $\kappa(X) = \delta(X)$, where $\delta(X)$ denotes the minimum degree of X.*

Lemma 2.2 $\kappa(P_n) = \delta(P_n) = n - 1$ *for any* $n \geq 3$.

Proof. By [21], we obtain that $g(P_n) = 6$, where $g(P_n)$ is the girth of P_n. So P_n is K_4-free, by Lemma 2.1, the assertion holds.■

In the following Lemma, we shall state some facts without proof. Some of these facts may be found in [21], and others follow immediately from the definition of the pancake graph.

Lemma 2.3 *Let $B^{(i)} = \{\pi \in S_n : \pi(1) = i\}$, $B_{(j)} = \{\pi \in S_n : \pi(n) = j\}$, $B^{(i)}_{(j)} = B^{(i)} \cap B_{(j)}$. Then the following (i)-(iii) hold:*

(i) For any $i \neq j$, each vertex in $B^{(i)}$ is adjacent to exactly one vertex in $B^{(j)}$;

(ii) For any $i \neq j$, each vertex in $B^{(i)}_{(j)}$ is adjacent to exactly one vertex in $B^{(j)}_{(i)}$ and exactly one vertex in $B^{(k)}_{(j)}$ for each $k \neq i, j$.

(iii) The mapping $\varphi : S_{n-1} \to B_{(j)}$ defined as $\varphi(\pi) = (j, n)\pi$ is an isomorphism from P_{n-1} to $P_n[B_{(j)}]$, where $P_n[B_{(j)}]$ is the subgraph of P_n induced by $B_{(j)}$.

Theorem 2.4 *If $n \geq 4$, then P_n is super-connected.*

Proof. It is enough to show that P_n contains no superatom. Suppose on the contrary that A is a superatom of P_n and consider the following possible cases:

Case 1. $A \subseteq B_{(i)}$ for some $i \in \{1, 2, \cdots, n\}$.

By Lemmas 2.2 and 2.3, we have $\kappa(P_n[B_{(i)}]) = \kappa(P_{n-1}) = n-2$ for $n-1 \geq 3$, so $|N(A) \cap B_{(i)}| \geq n-2$ for $n \geq 4$. Hence $|N(A)| = |N(A) \cap B_{(i)}| + |N(A) \cap (\bigcup_{j \neq i} B_{(j)})| \geq (n-2) + |A| \geq (n-2) + 2 = n > n-1 = \kappa(P_n)$, which is a contradiction.

Case 2. $A \nsubseteq B_{(i)}$ for any $i \in \{1, 2, \cdots, n\}$.

Then there exist i, j $(i \neq j)$ such that $A \cap B_{(i)} \neq \emptyset$ and $A \cap B_{(j)} \neq \emptyset$. Hence $|N(A)| \geq |N(A) \cap B_{(i)}| + |N(A) \cap B_{(j)}| \geq 2(n-2) \geq n-1 = \kappa(P_n)$, which is a contradiction. ■

Remark. If $n = 3$, then $P_3 = C_6$, clearly it is not super-connected.

Theorem 2.5 *If $n \geq 4$, then P_n is hyper-connected.*

Proof. By Theorem 2.4, P_n is super-connected for $n \geq 4$. Consider the vertex-transitivity of P_n, it suffices to show that $P_n - N[I]$ is connected, where $N[I]$ is the closed neighbourhood of the identity element I. We proceed by the induction on n. If $n = 4$, one can easily check that $P_4 - N[I]$ is connected.

If $n > 4$, then $P_n[B_{(i)}] - N[I]$ is connected for any $i < n$ since $|N[I] \cap B_{(1)}| = 1$ and $|N[I] \cap B_{(i)}| = 0$ for any $1 < i < n$. By induction, $P_n[B_{(n)}] - N[I] = P_{n-1} - N[I]$ is connected. By Lemma 2.3, each vertex in $B^{(n)}_{(i)}$ is adjacent to exactly one vertex in $B^{(i)}_{(n)}$ for any $i < n$. So for each $i < n$ there exists a vertex in $P_n[B_{(i)}] - N[I]$ which is adjacent to some vertex in $P_n[B_{(n)}] - N[I]$. Thus $P_n - N[I] = \bigcup_{i=1}^{n} P_n[B_{(i)}] - N[I]$ is connected. ■

Next we turn to consider the efficient dominating sets of P_n. By the definition of efficient dominating set, it is easy to see that any efficient dominating set D in P_n has $(n-1)!$ elements and $d(u, v) \geq 3$ for any $u, v \in D$, where $d(u, v)$ is the distance between two vertice u and v in P_n. Konstantinova in the abstract [7] obtained the following result on the efficient dominating set. For the completeness of this paper, here we present a proof of the result.

Theorem 2.6 [7] *There are exactly n efficient dominating sets $B^{(i)}$ $(1 \le i \le n)$ in P_n $(n \ge 3)$.*

Proof. Clearly each $B^{(i)}$ $(1 \le i \le n)$ is an efficient dominating set in P_n. So it suffices to prove that for any efficient dominating set D in P_n, if $D \cap B^{(i)} \ne \emptyset$, then $D = B^{(i)}$. Set $D^{(i)} = D \cap B^{(i)}$, $D_{(j)} = D \cap B_{(j)}$, $D^{(i)}_{(j)} = D \cap B^{(i)}_{(j)}$, $R^{(i)}_{(j)} = B^{(i)}_{(j)} \setminus D^{(i)}_{(j)}$. We consider the following cases:

Case 1. There exists a $D^{(i)}_{(j)}$ such that $D^{(i)}_{(j)} = B^{(i)}_{(j)}$.

By Lemma 2.3, $N(B^{(i)}_{(j)}) \cap B_{(i)} = B^{(j)}_{(i)}$, $N(N(B^{(i)}_{(j)})) \cap B_{(i)} = B_{(i)} \setminus B^{(j)}_{(i)}$. Since $B^{(i)}_{(j)} \subseteq D$ and $d(u,v) \ge 3$ for any $u, v \in D$, so we have $D \cap B^{(j)}_{(i)} = \emptyset$, $D \cap (B_{(i)} \setminus B^{(j)}_{(i)}) = \emptyset$, i.e. $D \cap B_{(i)} = \emptyset$. By Lemma 2.3 again, $N(B_{(i)}) = B^{(i)}$ and each vertex in $B_{(i)}$ is adjacent to exactly one vertex in $B^{(i)}$. Hence $B^{(i)} \subseteq D$. Since $|B^{(i)}| = |D| = (n-1)!$, we have $D = B^{(i)}$.

Case 2. There exists a $D^{(i)}_{(j)}$ such that $\emptyset \ne D^{(i)}_{(j)} \subsetneq B^{(i)}_{(j)}$.

By Lemma 2.3, we have $X_{(i)} := N(D^{(i)}_{(j)}) \cap B_{(i)} \subseteq B^{(j)}_{(i)}$, $Y_{(i)} := N(R^{(i)}_{(j)}) \cap B_{(i)} \cap D \subseteq B^{(j)}_{(i)}$, $Z_{(i)} := B^{(j)}_{(i)} \setminus (X_{(i)} \cup Y_{(i)})$, $W_{(i)} := N(Z_{(i)}) \cap D \subseteq B_{(i)} \setminus B^{(j)}_{(i)}$, $Y_{(j)} := N(R^{(i)}_{(j)}) \cap B_{(j)} \cap D \subseteq B_{(j)} \setminus B^{(i)}_{(j)}$. Now we claim that $D_{(i)} = Y_{(i)} \cup W_{(i)}$, $D_{(j)} = D^{(i)}_{(j)} \cup Y_{(j)}$. Clearly $D_{(i)} \supseteq Y_{(i)} \cup W_{(i)}$, $D_{(j)} \supseteq D^{(i)}_{(j)} \cup Y_{(j)}$. For any $x \in D_{(i)} \setminus Y_{(i)}$, then $x \in B_{(i)} \setminus (X_{(i)} \cup N(X_{(i)}) \cup Y_{(i)} \cup N(Y_{(i)}) \cup Z_{(i)}) = N(Z_{(i)}) \cap B_{(i)}$ and so $x \in N(Z_{(i)}) \cap B_{(i)} \cap D = W_{(i)}$. Hence $D_{(i)} \subseteq Y_{(i)} \cup W_{(i)}$. For any $y \in D_{(j)} \setminus D^{(i)}_{(j)}$, then $y \in B_{(j)} \setminus (D^{(i)}_{(j)} \cup N(D^{(i)}_{(j)}) \cup R^{(i)}_{(j)}) = N(R^{(i)}_{(j)}) \cap B_{(j)}$ and so $y \in N(R^{(i)}_{(j)}) \cap B_{(j)} \cap D = Y_{(j)}$. Hence $D_{(j)} \subseteq D^{(i)}_{(j)} \cup Y_{(j)}$.

Clearly $|X_{(i)}| = |D^{(i)}_{(j)}|$ and $|Y_{(i)}| + |Y_{(j)}| = |R^{(i)}_{(j)}|$, so $|X_{(i)}| + |Y_{(i)}| + |Y_{(j)}| = |B^{(i)}_{(j)}| = (n-2)!$. Since $|X_{(i)}| + |Y_{(i)}| + |Z_{(i)}| = |B^{(j)}_{(i)}| = (n-2)!$, we have $|W_{(i)}| = |Z_{(i)}| = |Y_{(j)}|$. By the definition of efficient dominating set and Lemma 2.3, for $k = i, j$, each vertex in $B_{(k)} \setminus (D_{(k)} \cup N(D_{(k)})$ is adjacent to exactly one vertex in $D^{(k)}$, each vertex in $D^{(k)}$ is adjacent to exactly one vertex in $B_{(k)} \setminus (D_{(k)} \cup N(D_{(k)})$. So $|D^{(i)}| = |B_{(i)} \setminus (D_{(i)} \cup N(D_{(i)})| = (n-1)! - (n-1)|D_{(i)}| = (n-1)! - (n-1)(|Y_{(i)}| + |W_{(i)}|) = (n-1)! - (n-1)((n-2)! - |X_{(i)}|) = (n-1)|X_{(i)}|$, $|D^{(j)}| = |B_{(j)} \setminus (D_{(j)} \cup N(D_{(j)})| = (n-1)! - (n-1)|D_{(j)}| = (n-1)! - (n-1)(|D^{(i)}_{(j)}| + |Y_{(j)}|) = (n-1)! - (n-1)((n-2)! - |Y_{(i)}|) = (n-1)|Y_{(i)}|$. Hence $|\bigcup_{k \ne i,j} D^{(i)}_{(k)}| = |D^{(i)}| - |D^{(i)}_{(j)}| = (n-1)|X_{(i)}| - |X_{(i)}| = (n-2)|X_{(i)}|$, $|\bigcup_{k \ne i,j} D^{(j)}_{(k)}| = |D^{(j)}| - |D^{(j)}_{(i)}| = (n-1)|Y_{(i)}| - |Y_{(i)}| = (n-2)|Y_{(i)}|$ and $|\bigcup_{k,l \ne i,j} D^{(l)}_{(k)}| = |D| - |D^{(i)}| - |D^{(j)}| - |W_{(i)}| - |Y_{(j)}| = (n-1)! - (n-1)|X_{(i)}| - (n-1)|Y_{(i)}| - |W_{(i)}| - |Y_{(j)}| = (n-1)((n-2)! - |X_{(i)}| - |Y_{(i)}|) - 2|Z_{(i)}| = (n-3)|Z_{(i)}|$.

By the definition of efficient dominating set and Lemma 2.3 , for any a fixed $l_0 \ne i, j$, each vertex in $\bigcup_{k \ne i,j} B^{(l_0)}_{(k)}$ either belongs to $\bigcup_{k \ne i,j} D_{(k)}$ or is adjacent to exactly one vertex in $\bigcup_{k \ne i,j} D_{(k)}$, each vertex in $\bigcup_{k \ne i,j} D_{(k)}$ either belongs to $\bigcup_{k \ne i,j} B^{(l_0)}_{(k)}$ or is adjacent to exactly one vertex in $\bigcup_{k \ne i,j} B^{(l_0)}_{(k)}$,so $(n-3)(n-2)! = |\bigcup_{k \ne i,j} B^{(l_0)}_{(k)}| = |\bigcup_{k \ne i,j} D_{(k)}| = |\bigcup_{k \ne i,j} D^{(i)}_{(k)}| + |\bigcup_{k \ne i,j} D^{(j)}_{(k)}| + |\bigcup_{k,l \ne i,j} D^{(l)}_{(k)}| = (n-2)|X_{(i)}| + (n-2)|Y_{(i)}| + (n-3)|Z_{(i)}| = (n-3)(|X_{(i)}| + |Y_{(i)}| + |Z_{(i)}|) + |X_{(i)}| + |Y_{(i)}| = (n-3)(n-2)! + |X_{(i)}| + |Y_{(i)}|$, hence $|D^{(i)}_{(j)}| = |X_{(i)}| = 0$, which is a contradiction. ■

3 The automorphism group of P_n

In this section, we completely determine the full automorphism group of P_n. First we introduce some definitions. Let $\mathrm{Sym}(\Omega)$ denote the set of all permutations of a set Ω. A *permutation representation* of a group G is a homomorphism from G into $\mathrm{Sym}(\Omega)$ for some set Ω. A permutation representation is also referred to as an action of G on the set Ω, in which case we say that G acts on Ω. Furthermore, if $\{g \in G : x^g = x, \forall x \in \Omega\} = 1$, we say the action of G on Ω is *faithful*, or G acts *faithfully* on Ω.

Theorem 3.1 *For $n \geq 5$, if $N(X) = B^{(i)}$ and $|X| = |B^{(i)}|$, where $X \subseteq V(P_n) = S_n$ and $i \in \{1, 2, \cdots, n\}$, then $X = B_{(i)}$.*

Proof. For $n = 5$, one can easily check that the assertion holds. We proceed by induction on n. First since $N(X) = \{y \in V(P_n) \setminus X : \exists\, x \in X,\ s.t.\ xy \in E(P_n)\}$, we have $X \cap N(X) = \emptyset$, i.e. $X \cap B^{(i)} = \emptyset$. Next we shall show that $X = B_{(i)}$ by the following three Claims:

Claim 1. Either $X = B_{(i)}$ or $X \cap B_{(i)} = \emptyset$.

Set $X_i := X \cap B_{(i)}$, $\overline{X_i} := X \setminus X_i$. Suppose on the contrary that $\emptyset \neq X_i \subsetneq B_{(i)}$. By Lemma 2.3 (iii), $P_n[B_{(i)}] \cong P_{n-1}$, so $P_n[B_{(i)}]$ is connected, which implies that $N(X_i) \cap B_{(i)} \neq \emptyset$. Since $N(X_i) \subseteq B_{(i)} \cup B^{(i)}$ and $\overline{X_i} \cap (B_{(i)} \cup B^{(i)}) = \emptyset$, we have $N(X_i) \cap \overline{X_i} = \emptyset$, i.e. $N(X_i) \subseteq N(X)$. So $N(X) \cap B_{(i)} \neq \emptyset$, which contradicts $N(X) = B^{(i)}$, hence Claim 1 holds.

Claim 2. Set $X_k = X \cap B_{(k)}$, $B_{(n-1)\to i,\, n\to k} = \{\pi \in S_n : \pi(n-1) = i, \pi(n) = k\}$. If $X \neq B_{(i)}$, then $X_k = B_{(n-1)\to i,\, n\to k}$ for any $k \neq i$.

By $X \neq B_{(i)}$ and Claim 1, $X \cap (B^{(i)} \cup B_{(i)}) = \emptyset$. By Lemma 2.3 (ii), $B^{(i)}_{(k)} \cap N(X_l) = \emptyset$ for any $k \neq l$. So we have $B^{(i)}_{(k)} \subseteq B^{(i)} = N(X) = N(\bigcup_{k\neq i} X_k) \subseteq \bigcup_{k \neq i} N(X_k) \Rightarrow B^{(i)}_{(k)} \subseteq N(X_k) \Rightarrow B^{(i)}_{(k)} \subseteq B_{(k)} \cap N(X_k)$. On the other hand, $B_{(k)} \cap N(X_k) \subseteq B_{(k)} \cap N(X) = B_{(k)} \cap B^{(i)} = B^{(i)}_{(k)}$. Thus $B_{(k)} \cap N(X_k) = B^{(i)}_{(k)}$. By Theorem 2.6, $B^{(i)}$ is an efficient dominating set of P_n, so $|X_k| \geq |B^{(i)}_{(k)}| \Rightarrow |X| = \sum_{k\neq i} |X_k| \geq \sum_{k \neq i} |B^{(i)}_{(k)}| = |B^{(i)}|$, note that $|X| = |B^{(i)}|$, and so $|X_k| = |B^{(i)}_{(k)}|$. By Lemma 2.3 (iii), $P_n[B_{(k)}] \cong P_{n-1}$ and $B^{(i)}_{(k)}$ is an efficient dominating set of $P_n[B_{(k)}]$. Since $B_{(k)} \cap N(X_k) = B^{(i)}_{(k)}$ and $|X_k| = |B^{(i)}_{(k)}|$, by induction, we have $X_k = B_{(n-1)\to i,\, n\to k}$, hence Claim 2 holds.

Claim 3. If $X \neq B_{(i)}$, then $n = 3$.

By $X \neq B_{(i)}$ and Claim 2, $X_k = B_{(n-1)\to i,\, n\to k}$ for any $k \neq i$. Since $X_k \subseteq B_{(k)}$ and $P_n[B_{(k)}] \cong P_{n-1}$, which is a $(n-2)$-regular graph, we have $|N(x_k) \cap B_{(k)}| = n - 2$ for any $x_k \in X_k$, note that $|N(x_k)| = n - 1$, and so $|N(x_k) \cap (\bigcup_{l \neq k} B_{(l)})| = 1$. Set $N(x_k) \cap (\bigcup_{l\neq k} B_{(l)}) = \{x_l\}$, where $x_l \in B_{(l)}$ for some $l \neq k, i$. Since $x_l \in N(x_k) \cap B_{(l)}$ and $N(x_k) \cap B_{(l)} \cap B^{(i)} = \emptyset$ (by Lemma 2.3), we have $x_l \notin B^{(i)}$. Note that $x_l \in N(x_k)$ and $N(X) = B^{(i)}$, then $x_l \in X_l = B_{(n-1)\to i,\, n\to l}$ (by Claim 2) and there exists a $r_{1j} \in PR_n$ such that $x_k = x_l r_{1j}$. Now we show that $j = n$. otherwise, we

have $j \neq n \Rightarrow r_{1j}(n) = n \Rightarrow k = x_k(n) = x_l r_{1j}(n) = x_l(n) = l$, which contradicts $k \neq l$. So $i = x_k(n-1) = x_l r_{1n}(n-1) = x_l(2) \Rightarrow n-1 = x_l^{-1}(i) = 2 \Rightarrow n = 3$, hence Claim 3 holds.

By Claim 3, if $X \neq B_{(i)}$, then $n = 3$, which contradicts $n \geq 5$. Hence $X = B_{(i)}$, the assertion holds. ■

Remark. For $n = 3, 4$, one can easily check that the result of Theorem 3.1 is not true. For example, in P_3, $N(\{(1\,2), (1\,3\,2)\}) = B^{(1)}$ and $|\{(1\,2), (1\,3\,2)\}| = |B_{(1)}| = 2$, however, $\{(1\,2), (1\,3\,2)\} \neq B_{(1)}$; In P_4, $N(\{(1\,2), (1\,2)(3\,4), (1\,3\,2), (1\,3\,4\,2), (1\,4\,2), (1\,4\,3\,2)\}) = B^{(1)}$ and $|\{(1\,2), (1\,2)(3\,4), (1\,3\,2), (1\,3\,4\,2), (1\,4\,2), (1\,4\,3\,2)\}| = |B_{(1)}| = 6$, however, $\{(1\,2), (1\,2)(3\,4), (1\,3\,2), (1\,3\,4\,2), (1\,4\,2), (1\,4\,3\,2)\} \neq B_{(1)}$.

Theorem 3.2 *If $n \geq 5$, then $\mathrm{Aut}(P_n) = L(S_n)$, where $L(S_n)$ is the left regular representation.*

Proof. For $n = 5$, a Nauty [18] computation shows that $|\mathrm{Aut}(P_5)| = 120$. Since $|\mathrm{Aut}(P_5)| \geq |L(S_5)| = 120$, we have $\mathrm{Aut}(P_5) = L(S_5)$. We proceed by induction on n. Clearly any automorphism of P_n must permute the efficient dominating sets of P_n. Let $\mathcal{B} = \{B^{(i)} : i = 1, 2, \cdots, n\}$. By Theorem 2.6, $\mathrm{Aut}(P_n)$ naturally acts on $\mathcal{B}$. Next we shall show that the action of $\mathrm{Aut}(P_n)$ on $\mathcal{B}$ is faithful. Assume that $\phi \in \mathrm{Aut}(P_n)$ such that $\phi(B^{(i)}) = B^{(i)}$ for each $i \in \{1, 2, \cdots, n\}$. By Lemma 2.3, $N(B_{(i)}) = B^{(i)}$, so we have $N(\phi(B_{(i)})) = \phi(B^{(i)}) = B^{(i)}$, $|\phi(B_{(i)})| = |B_{(i)}| = |B^{(i)}|$. By Theorem 3.1, $\phi(B_{(i)}) = B_{(i)}$ for each $i \in \{1, 2, \cdots, n\}$. Hence ϕ can be treated as an automorphism of $P_n[B_{(n)}] = P_{n-1}$, that is, the restriction $\phi\restriction B_{(n)} \in \mathrm{Aut}(P_{n-1}) = L(S_{n-1})$ by induction. For the identity element $I \in S_n$, set $y = \phi(I)$, then $y, I \in B^{(1)}_{(n)} \subseteq B_{(n)}$ and $\phi\restriction B_{(n)} = L(y)$. Hence

$$\begin{aligned} \phi(I) = y \quad &\Rightarrow \quad \phi(N(I) \cap B^{(i)}_{(n)}) = N(y) \cap B^{(i)}_{(n)} \\ &\Rightarrow \quad L(y)(r_{1,i}) = y r_{1,y^{-1}(i)} \\ &\Rightarrow \quad y r_{1,i} = y r_{1,y^{-1}(i)} \\ &\Rightarrow \quad y(i) = i, \end{aligned}$$

where $i = 2, 3, \cdots, n$. So we have $\phi(I) = y = I$, that is, ϕ fixes I. Since $\phi(B^{(i)}_{(j)}) = B^{(i)}_{(j)}$ for each $i, j \in \{1, 2, \cdots, n\}$, by Lemma 2.3 (ii) and the connectedness of P_n, ϕ fixes all vertice of P_n, so $\phi = 1$, which implies that the action of $\mathrm{Aut}(P_n)$ on $\mathcal{B}$ is faithful. Thus $Aut(P_n) \lesssim \mathrm{Sym}(\mathcal{B}) \Rightarrow |Aut(P_n)| \leq n!$. On the other hand, $|Aut(P_n)| \geq |L(S_n)| = n!$. Hence $Aut(P_n) = L(S_n)$. The assertion holds.■

Remark. If $n = 3$, then $P_3 = C_6$, so $\mathrm{Aut}(P_3) = D_{12}$, where D_{12} is the dihedral group of order 12. If $n = 4$, a Nauty computation shows that $|\mathrm{Aut}(P_4)| = 48$, so $L(S_4)$ is a normal subgroup of $\mathrm{Aut}(P_4)$. By Godsil [9], $\mathrm{Aut}(P_4)$ is the semiproduct $L(S_4) \rtimes \mathrm{Aut}(S_4, PR_4)$, where $\mathrm{Aut}(S_4, PR_4) = \{\phi \in \mathrm{Aut}(S_4) : \phi(PR_4) = PR_4\} = \{1, c((2\,3))\}$, here we denote by 1 the identity automorphism and by $c((2\,3))$ the automorphism induced by the conjugacy of $(2\,3)$ on S_4.

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